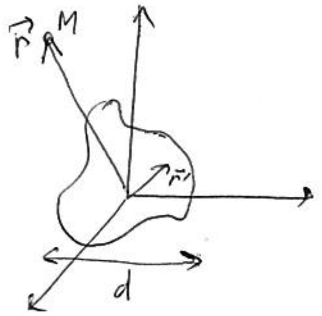
Потенцијали на великим растојањима



$r \gg r' \sim d$ - димензија области са наелектрисањима

$$\varphi(\vec r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r\,')dV'}{|\vec r-\vec r\,'|} \; ; \quad \frac{1}{|\vec r-\vec r\,'|} \text{ се развије у ред јер } r' \ll r$$

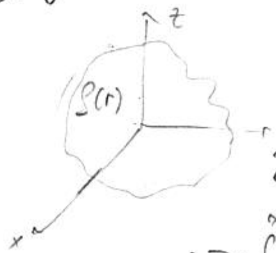
$$\frac{1}{|\vec r-\vec r\,'|} = \frac{1}{r}\left[1+\frac{\vec r\vec r\,'}{r^2}+\frac{3(\vec r\vec r\,')^2-r'^2r^2}{2r^4}+\dots\right]$$

$$\varphi(\vec r) = \frac{1}{4\pi\varepsilon_0}\Bigg[\frac{1}{r}\underbrace{\int\rho(\vec r\,')dV'}_{\substack{Q\\ \text{наелектрисање}}} + \frac{\vec r}{r^3}\underbrace{\int\rho(\vec r\,')\vec r\,'dV'}_{\vec p\text{-диполни момент}} + \frac{x_i x_j}{2r^5}\underbrace{\int\rho(\vec r\,')(3x_i'x_j'-\delta_{ij}r'^2)dV'}_{D_{ij}\text{ - квадруполни момент}}+\dots\Bigg]$$

1° $\mathrm{Tr}\,\hat D = \sum_i D_{ii} = \int\rho(\vec r)(3\underbrace{x_i x_i}_{r^2} - \underbrace{\delta_{ii}}_{3} r^2)\,dV = 0$

2° Ако систем има аксијалну симетрију $\hat D$ је дијагонална матрица

1. Запреминска густина наелектрисања електронског облака побуђеног водониковог атома је $\rho(\vec r) = -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta$, где је a Боров радијус, r растојање између протона и електрона. Одредити компоненте тензора $\hat D$, као и електрични диполни момент $\vec p$.



$$D_{ij} = \int\rho(\vec r)(3x_ix_j-\delta_{ij}r^2)dV \qquad x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta$$

$$D_{xx} = \int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta\,(3r^2\sin^2\theta\underbrace{\cos^2\varphi}_{\frac{1+\cos2\varphi}{2}\to 0} - r^2)\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= -2\pi k\int_0^\infty\int_0^\pi r^8 e^{-\frac{2r}{3a}}\sin^5\theta\left(\frac{3}{2}\sin^2\theta-1\right)dr\,d\theta$$

$$B(x,y) = 2\int_0^{\pi/2}(\sin\theta)^{2x-1}(\cos\theta)^{2y-1}d\theta = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)} \; ; \quad \Gamma(x) = \int e^{-t}t^{x-1}dt$$

$$\int_0^\pi\sin^7\theta\,d\theta = 2\int_0^{\pi/2}\sin^7\theta\,d\theta = B\left(4,\tfrac12\right) = \frac{\Gamma(4)\Gamma(\frac12)}{\Gamma(\frac92)} = \frac{3!\,\cancel{\sqrt\pi}}{\frac72\frac52\frac32\frac12\cancel{\sqrt\pi}} = \frac{\overset{2}{\cancel{6}}\cdot16}{7\cdot5\cdot\cancel{3}} = \frac{32}{35}$$

$$\int_0^\pi\sin^5\theta\,d\theta = 2\int_0^{\pi/2}\sin^5\theta\,d\theta = B\left(3,\tfrac12\right) = \frac{\Gamma(3)\Gamma(\frac12)}{\Gamma(\frac72)} = \frac{2\cancel{\sqrt\pi}}{\frac52\frac32\frac12\cancel{\sqrt\pi}} = \frac{16}{15}$$

или: $$\int_0^\pi\sin^5\theta\,d\theta = \int_{-1}^1(1-\cos^2\theta)^2 d\cos\theta = 2\int_0^1(1+\xi^4-2\xi^2)d\xi = 2\left(1+\frac15-\frac23\right) = \frac{16}{15}$$

$$\int_0^\pi\sin^7\theta\,d\theta = \int_{-1}^1(1-\cos^2\theta)^3 d\cos\theta = 2\int_0^1(1-3\xi^2+3\xi^4-\xi^6)d\xi = 2\left(\frac35-\frac17\right) = \frac{32}{35}$$

$$D_{xx} = -2\pi k\left(\frac32\frac{32}{35}-\frac{16}{15}\right)\int_0^\infty r^8e^{-\frac{2r}{3a}}dr = -2\pi k\frac{9\cdot\overset{16}{\cancel{32}}-16\cdot\overset{7}{\cancel{14}}}{\cancel{2}\cdot3\cdot5\cdot7}\left(\frac{3a}{2}\right)^9\Gamma(9) = \boxed{-2\pi k\frac{32}{105}\left(\frac{3a}{2}\right)^9 8!} = \boxed{-48\pi k(3a)^9}$$

$$D_{yy} = \int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4e^{-\frac{2r}{3a}}\sin^4\theta\,(3r^2\sin^2\theta\underbrace{\cos^2\varphi}_{\frac{1+\cos2\varphi}{2}\to0}-r^2)\,r^2\sin\theta\,dr\,d\theta\,d\varphi = D_{xx}$$

$$D_{zz} = \int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4e^{-\frac{2r}{3a}}\sin^4\theta\,(3r^2\underbrace{\cos^2\theta}_{1-\sin^2\theta}-r^2)\,r^2\sin\theta\,dr\,d\theta = \int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4e^{-\frac{2r}{3a}}\sin^4\theta\,(2r^2-3r^2\sin^2\theta)\,r^2\sin\theta\,dr\,d\theta = -2D_{xx}$$

$$D_{xy}=\int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta\; 3r^2\sin^2\theta\,\underbrace{\sin\varphi\cos\varphi}_{=0}\,\underbrace{r^2\sin\theta\, dr\, d\theta\, d\varphi}_{dV} = 0$$

$$D_{xz}=\int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta\; 3r^2\sin\theta\cos\theta\,\underbrace{\cos\varphi}_{=0}\, r^2\sin\theta\, dr\, d\theta\, d\varphi = 0$$

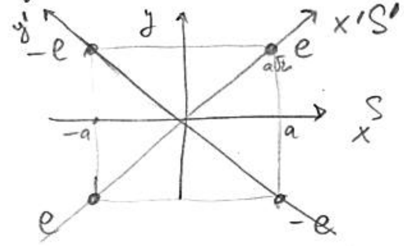
$$D_{yz}=\int_0^\infty\int_0^\pi\int_0^{2\pi} -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta\; 3r^2\sin\theta\cos\theta\,\underbrace{\sin\varphi}_{=0}\, r^2\sin\theta\, dr\, d\theta\, d\varphi = 0$$

$$\hat{D} = -2\pi k\, 8!\, \frac{32}{105}\left(\frac{3a}{2}\right)^9 \operatorname{diag}(1,1,-2)$$

$$\vec{p} = \int \rho\vec{r}\, dV = \int -kr^4 e^{-\frac{2r}{3a}}\sin^4\theta\,\left(r\sin\theta\cancel{\cos\varphi}\,\vec{e}_x + r\sin\theta\cancel{\sin\varphi}\,\vec{e}_y + r\cos\theta\,\vec{e}_z\right) r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$= -2\pi k\vec{e}_z\int_0^\infty r^7 e^{-\frac{2r}{3a}}\, dr \int_0^\pi \sin^5\theta\,\underbrace{\cos\theta\, d\theta}_{d\sin\theta} = -2\pi k\vec{e}_z\left(\frac{3a}{2}\right)^8\Gamma(8)\,\frac{\sin^6\theta}{6}\Big|_0^\pi = 0$$

2. У теменима квадрата странице $2a$ постављена су тачкаста наелектрисања супротних знакова. Одредити компоненте тензора $\hat{D}$ у системима S и S'.



$$D_{ij} = \sum_\alpha q_\alpha\left(3x_{\alpha i}x_{\alpha j} - \delta_{ij} r_\alpha^2\right)$$

У систему S:

$$D_{xx} = \sum_\alpha q_\alpha\left(3x_\alpha^2 - r_\alpha^2\right) = e(3a^2-2a^2) - e(3a^2-2a^2) + e(3a^2-2a^2) - e(3a^2-2a^2) = 0$$

$D_{yy} = \sum_\alpha q_\alpha\left(3y_\alpha^2 - r_\alpha^2\right) = 0$ $\quad D_{zz} = -D_{xx} - D_{yy} = 0$ $\quad D_{xz} = D_{yz} = 0$ (јер $z=0$ за сва наелектр.)

$$D_{xy} = \sum_\alpha q_\alpha 3x_\alpha y_\alpha = 3ea^2 - 3e(-a^2) + 3ea^2 - 3e(-a^2) = 12ea^2$$

$$\hat{D} = 12ea^2\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}$$

У систему S'

$$D'_{xx} = \sum_\alpha q_\alpha\left(3x_\alpha'^2 - r_\alpha'^2\right) = e(\underbrace{3\cdot 2a^2 - 2a^2}_{4a^2}) - e(\underbrace{0-2a^2}_{-2a^2}) + e(\underbrace{3\cdot 2a^2 - 2a^2}_{4a^2}) - e(\underbrace{0-2a^2}_{-2a^2}) = 12ea^2$$

$$D'_{yy} = \sum_\alpha q_\alpha\left(3y_\alpha'^2 - r_\alpha'^2\right) = e(0-2a^2) - e(3\cdot 2a^2 - 2a^2) + e(0-2a^2) - e(3\cdot 2a^2 - 2a^2) = -12ea^2$$

$D'_{zz} = -D'_{xx} - D'_{yy} = 0$ $\quad D'_{xz} = D'_{yz} = 0$ $\quad D'_{xy} = 0$ (увек је једна координата нула)

$$\hat{D}' = 12ea^2\begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix}$$

Провера: базис 1 $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\} \to \{\vec{e}_x{}', \vec{e}_y{}', \vec{e}_z{}'\}$ базис 2 $\quad \vec{e}_i{}' = R_{ij}\vec{e}_j \Rightarrow$

Вектор $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ се трансформише као $X \to X' = RX \quad X'_i = R_{ij}X_j$

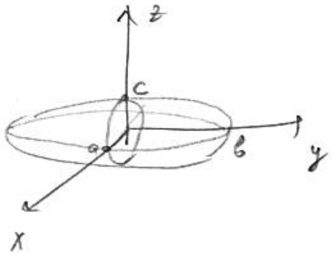
Матрица M се трансформише као $M \to M' = RMR^{-1}$ $\quad R$-ортогонална матрица

$\vec{e}_x{}' = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_y)$

$\vec{e}_y{}' = \frac{1}{\sqrt{2}}(-\vec{e}_x + \vec{e}_y)$ $\Rightarrow R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & \\ -1 & 1 & \\ & & \sqrt{2} \end{pmatrix}$; $\quad \hat{D}' = R\hat{D}R^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & \\ -1 & 1 & \\ & & \sqrt{2} \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & \end{pmatrix}\begin{pmatrix} 1 & -1 & \\ 1 & 1 & \\ & & \sqrt{2} \end{pmatrix}(12ea^2) = \frac{1}{2}12ea^2\begin{pmatrix} 1 & 1 & \\ -1 & 1 & \\ & & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 & \\ 1 & -1 & \\ & & \end{pmatrix}$

$$= \frac{1}{2}12ea^2\begin{pmatrix} 2 & & \\ & -2 & \\ & & \end{pmatrix} = \hat{D}'$$

3. Наелектрисање Q је распоређено по запремини елипсоида чије су полуосе a, b и c. У тачки $\vec{r}$ налази се диполни момент $\vec{d}$. Наћи ~~[illegible]~~ диполни и квадруполни момент система.



$\vec{p}=\int \rho\,\vec{r}\,dV$ уопштене сферне координате

$x = at\sin\theta\cos\varphi,\quad y = bt\sin\theta\sin\varphi,\quad z = ct\cos\theta$

$$J=\begin{vmatrix}\frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t}\\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta}\\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} & \frac{\partial z}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}a\sin\theta\cos\varphi & b\sin\theta\sin\varphi & c\cos\theta\\ at\cos\theta\cos\varphi & bt\cos\theta\sin\varphi & -ct\sin\theta\\ -at\sin\theta\sin\varphi & bt\sin\theta\cos\varphi & 0\end{vmatrix}$$

$$=abct^2\sin^3\theta\sin^2\varphi+abct^2\cos^2\theta\sin\theta\cos^2\varphi+abct^2\sin\theta\cos^2\theta\sin^2\varphi+abct^2\sin^3\theta\cos^2\varphi$$

$$=abct^2\sin^3\theta+abct^2\sin\theta\cos^2\theta=\boxed{abct^2\sin\theta}$$

$dV=abct^2\sin\theta\,dt\,d\theta\,d\varphi$; $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=t^2$ за елипсоида је $\boxed{t=1}$

$$V=\int_0^1\int_0^\pi\int_0^{2\pi} abc\,t^2\sin\theta\,dt\,d\theta\,d\varphi=abc\,\frac{1}{3}\,2\cdot 2\pi=\frac{4}{3}abc\pi \quad\Rightarrow\quad \rho=\frac{Q}{\frac{4}{3}abc\pi}=\frac{3Q}{4abc\pi}$$

$$\vec{p}=\int_0^1\int_0^\pi\int_0^{2\pi}\frac{3Q}{4abc\pi}\Big(\underbrace{at\sin\theta\cos\varphi\,\vec{e}_x}_{x\ \to 0}+\underbrace{bt\sin\theta\sin\varphi\,\vec{e}_y}_{y\ \to 0}+\underbrace{ct\cos\theta\,\vec{e}_z}_{z}\Big)abc\,t^2\sin\theta\,dt\,d\theta\,d\varphi$$

$$=\frac{3Q}{4\pi}\vec{e}_z\,2\pi\int_0^1\int_0^\pi ct^3\cos\theta\sin\theta\,dt\,d\theta=\frac{3Q}{2}\vec{e}_z\,c\,\frac{1}{4}\int_0^\pi\sin\theta\,d\sin\theta=0$$

$$D_{xx}=\int\rho\,(3x^2-r^2)\,dV=\int\rho\,(2x^2-y^2-z^2)\,dV$$

- $$\int\rho x^2dV=\rho\int_0^1\int_0^\pi\int_0^{2\pi}a^2t^2\sin^2\theta\underbrace{\cos^2\varphi}_{\frac{1+\cos2\varphi}{2}}\,abc\,t^2\sin\theta\,dt\,d\theta\,d\varphi=\rho\pi a^3bc\int_0^1\int_0^\pi t^4\sin^3\theta\,dt\,d\theta$$

$$=\rho\pi a^3bc\,\frac{1}{5}\int_{-1}^{1}(1-\xi^2)\,d\xi=\rho\pi a^3bc\,\frac{2}{5}\left(1-\frac{1}{3}\right)=\rho\pi a^3bc\,\frac{4}{15}$$

- $$\int\rho y^2dV=\rho\int_0^1\int_0^\pi\int_0^{2\pi}b^2t^2\sin^2\theta\underbrace{\sin^2\varphi}_{\frac{1+\cos2\varphi}{2}}\,abc\,t^2\sin\theta\,dt\,d\theta\,d\varphi=\rho b^3ac\pi\int_0^1\int_0^\pi t^4\sin^3\theta\,dt\,d\theta$$

$$=\rho b^3ac\pi\,\frac{4}{15}$$

- $$\int\rho z^2dV=\rho\int_0^1\int_0^\pi\int_0^{2\pi}c^2t^2\cos^2\theta\,abc\,t^2\sin\theta\,dt\,d\theta\,d\varphi=\rho\,2\pi abc^3\,\frac{1}{5}\left(-\frac{1}{3}\right)(-2)=\rho c^3ab\pi\,\frac{4}{15}$$

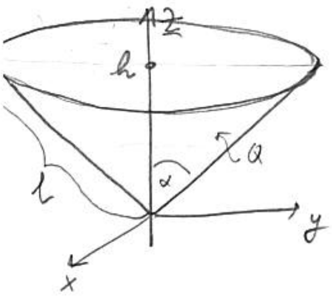
$$D_{xx}=\rho\pi\frac{4}{15}abc\,(2a^2-b^2-c^2)=\frac{3Q}{4abc\pi}\,\frac{4abc\pi}{15}\,(2a^2-b^2-c^2)=\boxed{\frac{Q}{5}(2a^2-b^2-c^2)}$$

$$\boxed{D_{yy}=\frac{Q}{5}(2b^2-a^2-c^2)}\qquad\boxed{D_{zz}=\frac{Q}{5}(2c^2-a^2-b^2)}$$

$$D_{xy}=\rho\int_0^1\int_0^\pi\int_0^{2\pi}3ab\,t^2\sin^2\theta\,\underbrace{\sin\varphi\cos\varphi}\,abc\,t^2\,dt\,d\theta\,d\varphi=0\qquad D_{yz}=D_{zx}=0$$

$$\hat{D}=\boxed{\frac{Q}{5}\,\mathrm{diag}\left(2a^2-b^2-c^2,\ 2b^2-a^2-c^2,\ 2c^2-a^2-b^2\right)}$$

4. Наелектрисање Q равномерно је распоређено по конусној површи која ротира угаоном брзином $\vec{\omega} = const.$ око z осе. Одредити векторски потенцијал и јачину магнетног поља на великим растојањима од система; угао конуса је $\alpha = 45^\circ$, а висина h.



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')dV'}{|\vec{r}-\vec{r}'|} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} + \ldots \quad \text{за } r' \ll r$$

$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{J}\,dV \rightsquigarrow \frac{1}{2}\int \vec{r}\times\vec{i}\,dS$$

$$\vec{i} = \sigma\vec{v} = \frac{Q}{S}\vec{\omega}\times\vec{r} = \frac{Q}{S}\omega\vec{e}_z\times(r\cos\alpha\,\overset{\nearrow 0}{\vec{e}_z} + r\sin\alpha\,\vec{e}_r) = \frac{Q\omega}{S}r\sin\alpha\,\vec{e}_\varphi$$

$$dS = r\sin\alpha\,d\varphi\,dr \Rightarrow S = \int_0^{2\pi}\int_0^{l} r\sin\alpha\,d\varphi\,dr = \frac{l^2}{2}\sin\alpha\,2\pi = l^2\pi\sin\alpha$$

$$\frac{h}{l} = \cos\alpha \Rightarrow l = \frac{h}{\cos\alpha} \Rightarrow \boxed{S = \frac{h^2\pi}{\cos^2\alpha}\sin\alpha} \Rightarrow \vec{i} = \frac{Q\omega\cancel{\sin\alpha}}{h^2\pi\cancel{\sin\alpha}}\cos^2\alpha\,r\vec{e}_\varphi$$

$$\boxed{\vec{i} = \frac{Q\omega\cos^2\alpha}{h^2\pi}r\vec{e}_\varphi}$$

$$\vec{m} = \frac{1}{2}\int\vec{r}\times\vec{i}\,dS = \frac{1}{2}\int_0^{l}\int_0^{2\pi}(r\cos\alpha\,\vec{e}_z + r\sin\alpha\,\vec{e}_r)\times\vec{e}_\varphi\,\frac{Q\omega\cos^2\alpha}{h^2\pi}r\cdot\underbrace{r\sin\alpha\,d\varphi\,dr}_{dS}$$

$$= \frac{Q\omega\cos^2\alpha\sin\alpha}{2h^2\pi}\int_0^{l}\int_0^{2\pi} r^3(-\cos\alpha\,\overset{\nearrow 0}{\vec{e}_r} + \sin\alpha\,\vec{e}_z)\,d\varphi\,dr = \frac{Q\omega\cos^2\alpha\sin^2\alpha}{\cancel{2\pi}h^2}\cancel{2\pi}\frac{l^4}{4}\vec{e}_z$$

$$= \frac{Q\omega\cos^2\alpha\sin^2\alpha}{4h^2}\frac{h^4}{\cos^4\alpha}\vec{e}_z = \frac{Q\omega h^2\sin^2\alpha}{4\cos^2\alpha}\vec{e}_z \quad \text{за } \alpha = \frac{\pi}{4} \quad \boxed{\vec{m} = \frac{Q\omega h^2}{4}\vec{e}_z}$$

површина конусне површи из интеграла $S = \frac{l^2\pi}{2l\pi}\cdot 2l\sin\alpha\,\pi = l^2\pi\sin\alpha$ ✓

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} = \frac{\mu_0}{4\pi}\frac{Q\omega h^2\sin^2\alpha}{4\cos^2\alpha\,r^3}\vec{e}_z\times(r\cos\theta\,\overset{\nearrow 0}{\vec{e}_z} + r\sin\theta\,\vec{e}_r) = \boxed{\frac{\mu_0 Q\omega h^2\sin^2\alpha\sin\theta}{16\pi\cos^2\alpha\,r^2}\vec{e}_\varphi}$$

$$\text{за } \alpha = 45^\circ \quad \boxed{\vec{A} = \frac{\mu_0 Q\omega h^2}{16\pi r^2}\sin\theta\,\vec{e}_\varphi} = C\cdot\frac{\sin\theta}{r^2}\vec{e}_\varphi$$

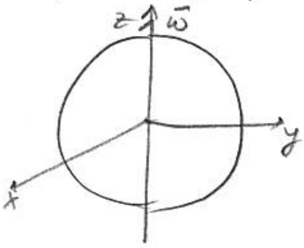
$$\vec{B} = \text{rot}\vec{A} = \frac{1}{r}\left\{\frac{1}{\sin\theta}\left(\frac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\vec{e}_r + \left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right)\vec{e}_\theta + \left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\vec{e}_\varphi\right\}$$

$$= \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\varphi)}{\partial\theta}\vec{e}_r - \frac{1}{r}\frac{\partial(rA_\varphi)}{\partial r}\vec{e}_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{C}{r^2}\sin^2\theta\right)\vec{e}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(C\frac{\sin\theta}{r}\right)\vec{e}_\theta$$

$$= \frac{\vec{e}_r\,C}{r^3\cancel{\sin\theta}}2\cancel{\sin\theta}\cos\theta + \frac{1}{r}C\sin\theta\frac{1}{r^2}\vec{e}_\theta = \frac{C}{r^3}(2\cos\theta\,\vec{e}_r + \sin\theta\,\vec{e}_\theta)$$

$$= \frac{\mu_0 Q\omega h^2}{16\pi r^3}(2\cos\theta\,\vec{e}_r + \sin\theta\,\vec{e}_\theta)\left(= \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m}\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)\right)$$

5. Наћи магнетни диполни момент равномерно наелектрисане сфере полупречника R и наелектрисања Q, ако сфера ротира угаоном брзином $\vec{\omega} = const$.



$$\sigma = \frac{Q}{4R^2\pi} \qquad \vec{i} = \sigma\vec{v} = \sigma\vec{\omega}\times\vec{r} = \sigma\omega\vec{e}_z\times R\vec{e}_r = \sigma\omega\vec{e}_z(R\cos\theta\,\vec{e}_z + R\sin\theta\,\vec{e}_{r_c})$$

$$\boxed{\vec{i} = \sigma\omega R\sin\theta\,\vec{e}_\varphi}$$

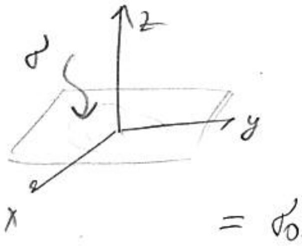
$$\vec{m} = \frac{1}{2}\int \vec{r}\times\vec{i}\,dS = \frac{1}{2}\int_0^\pi\int_0^{2\pi}(R\vec{e}_r\times\sigma\omega R\sin\theta\,\vec{e}_\varphi)R^2\sin\theta\,d\theta\,d\varphi$$

$$\vec{m} = \frac{1}{2}R^4\sigma\omega\int_0^\pi\int_0^{2\pi}-\vec{e}_\theta\sin^2\theta\,d\theta\,d\varphi = -\frac{R^4\sigma\omega}{2}\int_0^\pi\int_0^{2\pi}(-\sin\theta\,\vec{e}_z+\cos\theta\underbrace{\vec{e}_{r_c}}_{0})\sin^2\theta\,d\theta\,d\varphi$$

$$= \frac{R^4\sigma\vec{\omega}}{2}2\pi\int_0^\pi\sin^3\theta\,d\theta = \frac{R^4\sigma\vec{\omega}}{\cancel{2}}\cancel{2}\pi\int_{-1}^{1}(1-x^2)\,dx = 2\pi R^4\sigma\vec{\omega}\left(1-\frac{1}{3}\right) = \frac{4}{3}\pi R^4\sigma\vec{\omega}$$

$$\vec{m} = \frac{4}{3}\pi R^4\frac{Q}{4R^2\pi}\vec{\omega} = \boxed{\frac{QR^2}{3}\vec{\omega}}$$

6. Раван $z=0$ наелектрисана је површинском густином наелектрисања $\sigma = \sigma_0 e^{-\frac{r^2}{a^2}}$, где су σ_0 и a константе, r је растојање до z-осе. Одредити тензор квадруполног момента система.



$$D_{xx} = \int\sigma\,dS(3x^2-r^2) = \int\sigma\,dS(3x^2-x^2-y^2-\overset{0}{z^2}) = \int\sigma\,dS(2x^2-y^2)$$

$$= \int\sigma_0 e^{-\frac{r^2}{a^2}}\,r\,dr\,d\varphi\,(2r^2\underbrace{\cos^2\varphi}_{\frac{1+\cos2\varphi}{2}\to 0}-r^2\underbrace{\sin^2\varphi}_{\frac{1-\cos2\varphi}{2}\to 0}) = \int\sigma_0 e^{-\frac{r^2}{a^2}}r\,dr\,\frac{r^2}{\cancel{2}}\cancel{2}\pi$$

$$= \sigma_0\pi\int_0^\infty e^{-\frac{r^2}{a^2}}r^3\,dr \quad ; \quad t = \frac{r^2}{a^2} \Rightarrow dt = \frac{2r\,dr}{a^2} \; ; \; t\,dt = \frac{2r^3}{a^4}dr$$

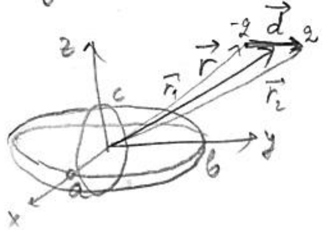
$$D_{xx} = \sigma_0\pi\int_0^\infty e^{-t}\frac{a^4}{2}t\,dt = \frac{\sigma_0\pi a^4}{2}\Gamma(2) = \frac{\sigma_0\pi a^4}{2} = D_{yy} \; ; \; D_{zz} = -\sigma_0\pi a^4$$

$$D_{xz} = D_{yz} = 0 \qquad D_{xy} = \int\sigma_0 e^{-\frac{r^2}{a^2}}r\,dr\,\underline{d\varphi}\;3r^2\underline{\sin\varphi\cos\varphi} = 0$$

$$\boxed{\hat{D} = \frac{\sigma_0\pi a^4}{2}\begin{pmatrix}1 & & \\ & 1 & \\ & & -2\end{pmatrix}}$$

Додатак из енергије:

(1) Наелектрисање Q је равномерно распоређено по запремини елипсоида чије су странице a, b, c, у $\vec{r}$ се налази диполни момент $\vec{d}$. Наћи енергију електростатичке интеракције дипола са елипсоидом урачунавајући квадруполни члан. Одредити и силу и момент силе који делују на дипол.



знамо: Q, $\vec{p}=0$; $\hat{D}=\frac{Q}{5}\,\mathrm{diag}\left(2a^2-b^2-c^2,\,2b^2-a^2-c^2,\,2c^2-a^2-b^2\right)$

$$\varphi=\frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r}+\frac{\vec{r}^{\,T}\hat{D}\vec{r}}{2r^5}\right]=\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r}+\frac{(2a^2-b^2-c^2)x^2+(2b^2-a^2-c^2)y^2+(2c^2-a^2-b^2)z^2}{10r^5}\right]$$

$$\vec{d}=q\vec{l}\ ;\quad W=q\left(\varphi(\vec{r}_2)-\varphi(\vec{r}_1)\right)=q\left(\varphi(\vec{r}_1+\vec{l})-\varphi(\vec{r}_1)\right)\approx q\left(\cancel{\varphi(\vec{r}_1)}+\vec{l}\nabla\varphi(\vec{r}_1)-\cancel{\varphi(\vec{r}_1)}\right)$$

$$W=\vec{d}\cdot\nabla\varphi(\vec{r})=-\vec{d}\cdot\vec{E}(\vec{r})$$

$$\nabla(r)=\vec{e}_i\,\partial_i\underbrace{(x_jx_j)}_{r^2}{}^{1/2}=\vec{e}_i\,\frac{1}{\cancel{2}}\frac{1}{r}\cdot\cancel{2}x_i=\frac{\vec{r}}{r}\ ;\quad \nabla\left(\frac{1}{r}\right)=-\frac{1}{r^2}\cdot\nabla r=-\frac{1}{r^3}\vec{r}$$

$$\nabla\left(\frac{\vec{r}^{\,T}\hat{D}\vec{r}}{r^5}\right)=-5\,\frac{1}{r^6}\cdot\frac{\vec{r}}{r}\cdot(\vec{r}^{\,T}\hat{D}\vec{r})+\frac{1}{r^5}\,\vec{e}_i\,\partial_i\left(x_jD_{jk}x_k\right)$$

$$=-\frac{5}{r^7}\vec{r}\,(\vec{r}^{\,T}\hat{D}\vec{r})+\frac{1}{r^5}\vec{e}_i\left(D_{ik}x_k+D_{ji}x_j\right)=-\frac{5}{r^7}\vec{r}\,(\vec{r}^{\,T}\hat{D}\vec{r})+\frac{2\hat{D}\vec{r}}{r^5}$$

$$W=\frac{1}{4\pi\varepsilon_0}\vec{d}\cdot\nabla\left(\frac{Q}{r}+\frac{\vec{r}^{\,T}\hat{D}\vec{r}}{2r^5}\right)=\frac{1}{4\pi\varepsilon_0}\vec{d}\left(-\frac{Q}{r^3}\vec{r}-\frac{5}{2r^7}\vec{r}\,(\vec{r}^{\,T}\hat{D}\vec{r})+\frac{\hat{D}\vec{r}}{r^5}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\left(-\frac{Q(\vec{d}\cdot\vec{r})}{r^3}-\frac{5(\vec{d}\cdot\vec{r})}{2r^7}\frac{Q}{5}\left((2a^2-b^2-c^2)x^2+(2b^2-a^2-c^2)y^2+(2c^2-a^2-b^2)z^2\right)+\frac{Q}{5r^5}\left[(2a^2-b^2-c^2)d_xx+(2b^2-c^2-a^2)d_yy+(2c^2-a^2-b^2)d_zz\right]\right)$$

$$\vec{F}=-q\vec{E}(\vec{r}_1)+q\vec{E}(\vec{r}_2)=-q\cancel{\vec{E}(\vec{r}_1)}+q\left(\cancel{\vec{E}(\vec{r}_1)}+(\vec{l}\nabla)E(\vec{r}_1)\right)=q(\vec{l}\nabla)\vec{E}(\vec{r})=(\vec{d}\nabla)\vec{E}(\vec{r})$$

$$\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\left(\frac{Q\vec{r}}{r^3}+\frac{5}{2r^7}\vec{r}\,(\vec{r}^{\,T}\hat{D}\vec{r})-\frac{\hat{D}\vec{r}}{r^5}\right)=$$

$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{\vec{r}}{r^3}+\frac{\vec{r}\left((2a^2-b^2-c^2)x^2+(2b^2-a^2-c^2)y^2+(2c^2-a^2-b^2)z^2\right)}{2r^7}-\frac{(2a^2-b^2-c^2)x+(2b^2-a^2-c^2)y+(2c^2-a^2-b^2)z}{5r^5}\right)$$

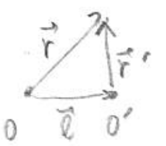
$$\vec{F}=(d_x\partial_x+d_y\partial_y+d_z\partial_z)E(\vec{r})$$

$$\vec{M}=\vec{r}_1\times(-q\vec{E}(\vec{r}_1))+\vec{r}_2\times q\vec{E}(\vec{r}_2)=\vec{r}_1\times(-q\vec{E}(\vec{r}_1))+(\vec{r}_1+\vec{l})\times q\left(\vec{E}(\vec{r}_1)+(\vec{l}\nabla)\vec{E}(\vec{r}_1)\right)$$

$$=\cancel{-q\vec{r}_1\times\vec{E}(\vec{r}_1)}+\cancel{q\vec{r}_1\times\vec{E}(\vec{r}_1)}+q\vec{l}\times\vec{E}(\vec{r}_1)+q\vec{r}_1\times(\vec{l}\nabla)\vec{E}(\vec{r}_1)+\underbrace{q\vec{l}\times(\vec{l}\nabla)\vec{E}(\vec{r}_1)}_{\vec{d}\text{ мало}}{}^{0}$$

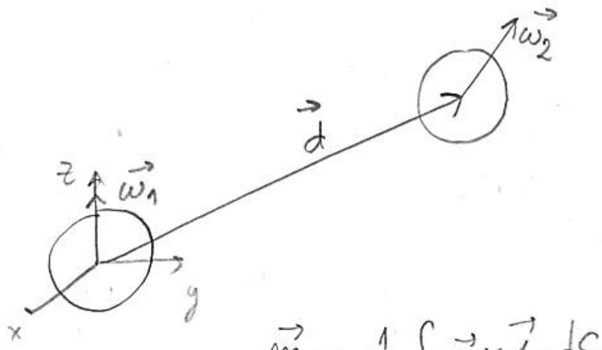
$$=\vec{d}\times\vec{E}(\vec{r})+\vec{r}\times\underbrace{\left((\vec{d}\nabla)\vec{E}(\vec{r})\right)}_{\vec{F}}=\vec{d}\times\vec{E}(\vec{r})+\vec{r}\times\vec{F}$$

ово је момент силе у односу на центар дипола.



$$\vec{M}_0=\vec{r}\times\vec{F}=(\vec{l}+\vec{r}')\times\vec{F}=\vec{l}\times\vec{F}+\vec{r}'\times\vec{F}=\vec{l}\times\vec{F}+\vec{M}_{0'}$$

Две сфере једнаког полупречника R и једнаког укупног наелектрисања Q, равномерно распоређеног по њиховим површинама, са центрима на међусобном растојању d $(d>>R)$ ротирају константним угаоним брзинама $\vec\omega_1$ и $\vec\omega_2$. Одредити енергију интеракције њихових магнетних поља. Одредити сопствену енергију магнетног поља сваке од сфера.



$$W_{int} = \frac{1}{\mu_0}\int_{\text{цео пр.}}\vec B_1(\vec r)\cdot\vec B_2(\vec r)\,dV = \ldots = \int_{V_1}\vec J_1(\vec r)\cdot\vec A_2(\vec r)\,dV$$

$$= \int_{V_2}\vec J_2(\vec r)\cdot\vec A_1(\vec r)\,dV$$ $\vec A_1$ развијемо око неке тачке у области V_2

$$= \ldots = \vec m_2\cdot\vec B_1$$

$$\vec m_1 = \frac12\int_{S_1}\vec r\times\vec\imath_1\,dS = \frac12\int_{S_1}\vec r\times\left(\frac{Q}{4R^2\pi}\vec\omega_1\times\vec r\right)dS = \frac{Q}{8R^2\pi}\int\left(\vec\omega_1R^2-\vec r(\vec r\cdot\vec\omega_1)\right)dS$$

$$= \frac{Q}{8R^2\pi}\left[\vec\omega_1R^2\cdot4R^2\pi - \int_0^\pi\int_0^{2\pi}\left(R\cos\theta\,\vec e_z + \cancelto{0}{R\sin\theta\,\vec e_{r_c}}\right)R\cos\theta\,\omega_1\cdot\underbrace{R^2\sin\theta\,d\theta\,d\varphi}_{dS}\right]$$

$$= \frac{Q}{8R^2\pi}\left[4R^4\pi\,\vec\omega_1 - R^4\vec\omega_1\,2\pi\,\frac23\right] = \frac23\frac{QR^2}{2}\vec\omega_1 = \frac13 QR^2\vec\omega_1$$

слично $\vec m_2 = \frac13 QR^2\vec\omega_2$, па је:

$$W_{int} \approx \vec m_2\cdot\vec B_1(\vec d)\quad;\quad \vec B_1(\vec d) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec m_1\vec d)\vec d}{d^5} - \frac{\vec m_1}{d^3}\right)$$ у диполној апроксим.

$$W_{int} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec m_1\vec d)(\vec m_2\vec d)}{d^5} - \frac{\vec m_1\cdot\vec m_2}{d^3}\right)$$

$$= \boxed{\frac{\mu_0}{4\pi}\left(\frac{QR^2}{3}\right)^2\left(\frac{3(\vec\omega_1\vec d)(\vec\omega_2\vec d)}{d^5} - \frac{\vec\omega_1\cdot\vec\omega_2}{d^3}\right)}$$

$$W_{соп.} = \frac{1}{2\mu_0}\int\vec B^2\,dV = \ldots = \frac12\int\vec J(\vec r)\cdot\vec A(\vec r)\,dV = \frac12\int\vec\imath(\vec r)\cdot\vec A(\vec r)\,dS$$

Треба да нађемо $\vec A$ на сфери

$$\Delta\vec A = -\mu_0\vec J = -\mu_0\frac{Q}{4R^2\pi}(\vec\omega\times\vec r)\delta(r-R) = -\mu_0\frac{Q}{4R^2\pi}\omega R\sin\theta\,\vec e_\varphi\,\delta(r-R)$$

$$\vec A = f(r)\cdot\sin\theta\,\vec e_\varphi$$

$$\Delta\vec A = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)f(r)\sin\theta\,\vec e_\varphi = \left(f'' + \frac2r f' - \frac{2}{r^2}f\right)\sin\theta\,\vec e_\varphi$$

$f\sim r^\alpha;\ \alpha = 1$ или

$$\vec A_< = C_1\cdot r\cdot\sin\theta\,\vec e_\varphi \qquad \vec A_> = \frac{C_2}{r^2}\sin\theta\,\vec e_\varphi \qquad \vec A_< = \vec A_>|_{r=R} \qquad \vec e_r\times(\mathrm{rot}\vec A_> - \mathrm{rot}\vec A_<)| \underset{\varphi}{=} \mu_0\frac{Q}{4R^2\pi}\omega R\sin\theta\,\vec e_\varphi$$

$$\boxed{C_1R = \frac{C_2}{R^2}}\ ;\quad \vec e_r\times\left(-\frac1r\frac{\partial(rf_>)}{\partial r}\sin\theta\,\vec e_\theta + \frac1r\frac{\partial(rf_<)}{\partial r}\sin\theta\,\vec e_\theta\right)\Big|_{r=R} = \frac{\mu_0Q}{4R\pi}\omega\sin\theta\,\vec e_\varphi$$

$$-\frac1R\left(f_>(R)+Rf'_>(R)\right)\sin\theta\,\vec e_\varphi + \frac1R\left(f_<(R)+Rf'_<(R)\right)\sin\theta\,\vec e_\varphi = \frac{\mu_0Q\omega}{4R\pi}\sin\theta\,\vec e_\varphi \Rightarrow -f'_>(R)+f'_<(R) = \frac{\mu_0Q\omega}{4R\pi}$$

$$\boxed{\frac{2C_2}{R^3} + C_1 = \frac{\mu_0Q\omega}{4R\pi}} \Rightarrow \frac{3C_2}{R^3} = \frac{\mu_0Q\omega}{4R\pi} \Rightarrow C_2 = \frac{\mu_0Q\omega R^2}{12\pi} \qquad C_1 = \frac{\mu_0Q\omega}{12\pi R}$$

$$\vec{A}_< = \frac{\mu_0 Q\omega}{12\pi R} r\sin\theta\,\vec{e}_\varphi \qquad \vec{A}_> = \frac{\mu_0 Q\omega R^2}{12\pi r^2}\sin\theta\,\vec{e}_\varphi$$

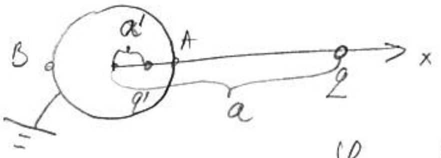
$$\vec{A}(R) = \frac{\mu_0 Q\omega}{12\pi}\sin\theta\,\vec{e}_\varphi$$

$$W_{\text{coū},1} = \frac{1}{2}\int_{S_1} \vec{\imath}_1(\vec{R})\cdot\vec{A}_1(\vec{R})\,dS = \frac{1}{2}\int \underbrace{\frac{Q}{4R^2\pi}\,\omega_1 R\sin\theta\,\vec{e}_\varphi}_{\vec{\imath}_1(\vec{R})}\cdot\frac{\mu_0 Q\omega_1}{12\pi}\sin\theta\,\vec{e}_\varphi\cdot R^2\sin\theta\,d\theta\,d\varphi$$

$$= \frac{1}{2}\frac{\mu_0 Q^2\omega_1^2 R}{48\pi^2}\int_0^{\pi}\int_0^{2\pi}\underbrace{\sin^3\theta\,d\theta}_{-(1-\cos^2\theta)\,d\cos\theta}\,d\varphi = \frac{\mu_0 Q^2\omega_1^2 R}{96\pi^{\cancel{2}}}\cdot 2\cancel{\pi}\cdot\left(2-\frac{2}{3}\right) = \frac{\mu_0 Q^2\omega_1^2 R}{96\pi}\,\frac{8}{3} = \boxed{\frac{\mu_0 Q^2\omega_1^2 R}{36\pi}}$$

$$\boxed{W_{\text{coū},2} = \mu_0\frac{Q^2\omega_2^2 R}{36\pi}}$$

Наћи енергију U и силу $\vec{F}$ интеракције тачкастог наелектрисања q са уземљеном проводном сфером радијуса R. Наелектрисање се налази на растојању a од центра сфере. Систем се налази у хомогеном диелектрику, диелектричне пропустљивости ε.



Потенцијал сфере је нула. Лик се налази на растојању x од центра кугле.

$$\varphi_A = \frac{1}{4\pi\varepsilon}\left(\frac{q}{a-R} + \frac{q'}{R-x}\right) = 0 \qquad \varphi_B = \frac{1}{4\pi\varepsilon}\left(\frac{q}{a+R} + \frac{q'}{a'+R}\right) = 0$$

$$q' = -\frac{q(R-a')}{a-R} = -\frac{q(R+a')}{a+R}$$

$$(a+R)(R-a') = (a-R)(R+a')$$

$$\cancel{Ra} + R^2 - aa' + \cancel{Ra'} = \cancel{aR} - R^2 + aa' + \cancel{Ra'} \Rightarrow aa' = R^2, \ \boxed{a' = \frac{R^2}{a}}$$

$$q' = -\frac{q\left(R - \frac{R^2}{a}\right)}{a-R} = -\frac{q\frac{R}{a}(a-R)}{a-R} = \boxed{-q\frac{R}{a}}$$

$$\vec{F} = \frac{1}{4\pi\varepsilon}\frac{q\cdot q'}{(a-a')^2}\vec{e}_x = \frac{1}{4\pi\varepsilon}\frac{-q^2 R/a}{\left(a - \frac{R^2}{a}\right)^2}\vec{e}_x = \boxed{-\frac{1}{4\pi\varepsilon}\frac{q^2 R a}{(a^2-R^2)^2}\vec{e}_x}$$

$$\vec{F} = -\frac{dU}{dx}\vec{e}_x \Rightarrow \int_a^\infty dU = -\int_a^\infty \vec{F}\,dx\,\vec{e}_x$$

x растојање наелектрисања и центра сфере

$$\overset{0}{U_\infty} - U_a = -\int_a^\infty \vec{F}\vec{e}_x\,dx \Rightarrow U_a = \int_0^\infty \vec{F}\vec{e}_x\,dx = A$$

рад на премештање из a у ∞ наелектрисања q.

$$A = \int_a^\infty\left(-\frac{1}{4\pi\varepsilon}\right)\frac{q^2 R x}{(x^2-R^2)^2}dx = -\frac{1}{4\pi\varepsilon}\int_a^\infty \frac{q^2 R\,d(x^2-R^2)}{(x^2-R^2)^2}\,\frac{1}{2} = -\frac{1}{8\pi\varepsilon}q^2 R(-1)\frac{1}{x^2-R^2}\Big|_a^\infty$$

$$U_a = A = \frac{1}{8\pi\varepsilon}q^2 R(-1)\frac{1}{a^2-R^2} = \boxed{\frac{-q^2 R}{8\pi\varepsilon(a^2-R^2)}}$$

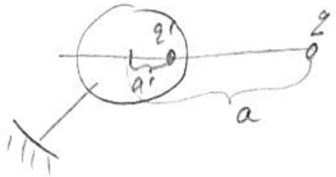
II начин: подсетник: $1^\circ \ w = \frac{1}{2}\varepsilon_0\vec{E}^2 \Rightarrow W_{сопст.} = \frac{1}{2}\int \varepsilon_0 \vec{E}^2 dV = \frac{1}{2}\int \rho\varphi\,dV$

↑ потенцијал који генерише

$2^\circ \ \vec{E}_1 + \vec{E}_2 = \vec{E} \qquad W = \frac{1}{2}\int \varepsilon_0(\vec{E}_1+\vec{E}_2)^2 dV = \underbrace{\frac{1}{2}\varepsilon_0\int \vec{E}_1^2 dV}_{W_{1сопст.}} + \underbrace{\frac{1}{2}\varepsilon_0\int \vec{E}_2^2 dV}_{W_{2сопст.}} + \underbrace{\varepsilon_0\int \vec{E}_1\vec{E}_2\,dV}_{W_{int}}$

$$W_{int} = \varepsilon_0\int \rho_1\varphi_2\,dV$$

↑ потенцијал генерише ρ_2, ρ_1 и ρ_2 две различите расподеле



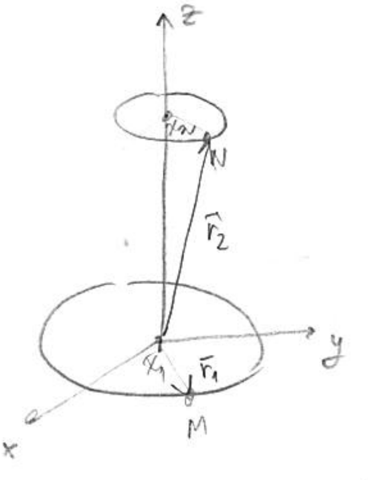
$q' = -q\frac{R}{a}$; $a' = \frac{R^2}{a}$ $\quad \varphi' = \frac{1}{4\pi\varepsilon}\frac{q'}{r'}$ $\quad$ const.

Изгледа као $U = q\cdot\varphi'$ међутим $\varphi' = f(q) = q\cdot\boxed{\frac{1}{4\pi\varepsilon}\frac{-\frac{R}{a}}{a-\frac{R^2}{a}}} = \alpha\cdot q$

$dU = dq\cdot\alpha\cdot q$ ← енергија интеракције дела наелектрисања q са ликом!

$$U = \int dq\,\alpha\cdot q = \alpha\cdot\frac{q^2}{2} = -\frac{R}{4\pi\varepsilon(a^2-R^2)}\frac{q^2}{2} = \boxed{-\frac{q^2 R}{8\pi\varepsilon(a^2-R^2)}}$$

Два кружна проводника полупречника a и b леже у равнима које су међусобно паралелне тако да је права која спаја њихове центре нормална на те равни. Растојање између центара износи h. Одредити коефицијент узајамне индукције ових проводника. Посебно размотрити случај када је растојање међу проводницима веома велико $h >> \sqrt{a^2+b^2}$



Сопствена енергија за систем сталних квазилинијских струја

$$W_{сот} = \frac{1}{2}\int \vec{j}_{ук}(\vec{r})\cdot \vec{A}_{ук}(\vec{r}) = \frac{\mu_0}{8\pi}\sum_{i,k=1}^{N}\int \frac{\vec{j}_i(\vec{r})\cdot \vec{j}_k(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV\,dV'$$

$$W_{ik} = \frac{\mu_0}{4\pi}\int \frac{\vec{j}_i(\vec{r})\cdot \vec{j}_k(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,dV\,dV' = \frac{\mu_0}{4\pi}\int \frac{I_i\cdot I_k\, d\vec{r}\,d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}$$

$$= I_i I_k\cdot \frac{\mu_0}{4\pi}\int \frac{d\vec{r}\,d\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = I_i I_k L_{ik}$$

$$L = \frac{\mu_0}{4\pi}\int \frac{d\vec{r}_1\,d\vec{r}_2}{|\vec{r}_1-\vec{r}_2|}$$

$$\vec{r}_1 = a\cos x_1\,\vec{e}_x + a\sin x_1\,\vec{e}_y;\quad d\vec{r}_1 = (-a\sin x_1\,\vec{e}_x + a\cos x_1\,\vec{e}_y)\,dx_1$$

$$\vec{r}_2 = b\cos x_2\,\vec{e}_x + b\sin x_2\,\vec{e}_y + h\,\vec{e}_z;\quad d\vec{r}_2 = (-b\sin x_2\,\vec{e}_x + b\cos x_2\,\vec{e}_y)\,dx_2$$

$$d\vec{r}_1\cdot d\vec{r}_2 = ab(\sin x_1\sin x_2 + \cos x_1\cos x_2)\,dx_1\,dx_2 = ab\cos(x_2-x_1)\,dx_1\,dx_2$$

$$|\vec{r}_1-\vec{r}_2| = \left((a\cos x_1 - b\cos x_2)^2 + (a\sin x_1 - b\sin x_2)^2 + h^2\right)^{1/2} = \left(a^2+b^2+h^2-2ab\cos(x_2-x_1)\right)^{1/2}$$

$$L = \frac{\mu_0}{4\pi}\int_0^{2\pi}\!\!\int_0^{2\pi}\frac{ab\cos(x_2-x_1)\,dx_1\,dx_2}{\left(a^2+b^2+h^2-2ab\cos(x_2-x_1)\right)^{1/2}} = \frac{\mu_0}{4\pi}\int_0^{2\pi}dx_2\int_0^{2\pi}\frac{ab\cos(x_2-x_1)\,dx_1}{\left(h^2+a^2+b^2-2ab\cos(x_1-x_2)\right)^{1/2}}\quad \boxed{x_1 - x_2 = \alpha \text{ смена}}$$

$$\int_0^{2\pi}\frac{ab\cos(x_1-x_2)\,dx_1}{\left(h^2+a^2+b^2-2ab\cos(x_1-x_2)\right)^{1/2}} = \int_{-x_2}^{2\pi-x_2}\frac{ab\cos\alpha\,d\alpha}{\left(h^2+a^2+b^2-2ab\cos\alpha\right)^{1/2}} = \int_0^{2\pi}\frac{ab\cos\alpha\,d\alpha}{\left(h^2+a^2+b^2-2ab\cos\alpha\right)^{1/2}}$$

не зависи од x_2

$$L = \frac{\mu_0}{2}\cdot ab\int_0^{2\pi}\frac{\cos\alpha\,d\alpha}{\left(h^2+a^2+b^2-2ab\cos\alpha\right)^{1/2}} = \mu_0 ab\int_0^{\pi}\frac{\cos\alpha\,d\alpha}{\left(h^2+a^2+b^2-2ab\cos\alpha\right)^{1/2}}$$

јер $\int_0^{2\pi} f(\cos x)\,dx = 2\int_0^{\pi} f(\cos x)\,dx$

$\cos(\pi - x) = -\cos x$

смена $\alpha = \pi - 2\beta$; $d\alpha = -2\,d\beta$

$$L = -2\mu_0 ab\int_{\pi/2}^{0}\frac{\cos(\pi-2\beta)\,d\beta}{\left(h^2+a^2+b^2-2ab\cos(\pi-2\beta)\right)^{1/2}} = -2\mu_0 ab\int_0^{\pi/2}\frac{\cos(2\beta)\,d\beta}{\left(h^2+a^2+b^2+2ab\cos 2\beta\right)^{1/2}}$$

$$\cos 2\beta = \cos^2\beta - \sin^2\beta = 1 - 2\sin^2\beta$$

$$L = -2\mu_0 ab\int_0^{\pi/2}\frac{(1-2\sin^2\beta)\,d\beta}{\left(h^2+a^2+b^2+2ab(1-2\sin^2\beta)\right)^{1/2}} = -2\mu_0 ab\int_0^{\pi/2}\frac{(1-2\sin^2\beta)\,d\beta}{\left(h^2+(a+b)^2-4ab\sin^2\beta\right)^{1/2}}$$

извучемо из корена

$$= -\frac{2\mu_0 ab}{\sqrt{h^2+(a+b)^2}}\int_0^{\pi/2}\frac{1-2\sin^2\beta}{\left(1-k^2\sin^2\beta\right)^{1/2}},\quad k^2 = \frac{4ab}{h^2+(a+b)^2};\quad k = \frac{2\sqrt{ab}}{\sqrt{h^2+(a+b)^2}}$$

$$\boxed{K(k) = \int_0^{\pi/2}\frac{d\beta}{(1-k^2\sin^2\beta)^{1/2}};\quad E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta}\,d\beta}$$ елиптички интеграли

$$L = -\mu_0\sqrt{ab}\,k\int_0^{\pi/2}\frac{1+\frac{2}{k^2}\left(-k^2\sin^2\beta+1\right)-\frac{2}{k^2}}{\left(1-k^2\sin^2\beta\right)^{1/2}} = -\mu_0\sqrt{ab}\,k\left[\left(1-\frac{2}{k^2}\right)K(k)+\frac{2}{k^2}E(k)\right]$$

$$= \boxed{\mu_0\sqrt{ab}\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right]}$$

Ако је $h^2 >> a^2+b^2$ важиће: $k^2 \approx \frac{4ab}{h^2} << 1$

$$K(k) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} \approx \int_0^{\pi/2} d\beta \left[1 - \frac{1}{2}(-k^2\sin^2\beta) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2} k^4 \sin^4\beta, \underbrace{- \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{6} k^6 \sin^6\beta + \ldots}_{\text{не треба}}\right]$$

$(1+x)^\alpha \approx 1 + \alpha x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \ldots$ за $x<<1$, $x = -k^2\sin^2\beta$; $\alpha = -\frac{1}{2}$

$$K(k) \approx \int_0^{\pi/2} d\beta \left[1 + \frac{k^2}{2}\sin^2\beta + \frac{3}{8}k^4\sin^4\beta, + \frac{5}{16}k^6\sin^6\beta\right] = \frac{\pi}{2} + \frac{k^2}{2}\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right) + \frac{3}{8}k^4\frac{1}{2}B\left(\frac{5}{2},\frac{1}{2}\right) + \underbrace{\frac{5}{16}k^6\frac{1}{2}B\left(\frac{7}{2},\frac{1}{2}\right)}_{\text{не треба}}$$

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}t \, \cos^{2y-1}t \, dt \qquad \Gamma(x+1) = \Gamma(x)\cdot x$$

$$K(k) \approx \frac{\pi}{2} + \frac{k^2}{2}\frac{1}{2}\frac{\frac{1}{2}\pi}{1} + \frac{3}{8}k^4\frac{1}{2}\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}^2}{2} + \frac{5}{16}k^6\frac{1}{2}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{3\cdot 2} = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \left(\frac{3}{8}\right)^2 k^4 + \left(\frac{5}{16}\right)^2 k^6\right)$$

$$E(k) = \int_0^{\pi/2}\sqrt{1-k^2\sin^2\beta} \approx \int_0^{\pi/2} d\beta \left[1 + \frac{1}{2}(-k^2\sin^2\beta) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}k^4\sin^4\beta + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\cdot 2}(-k^6\sin^6\beta) + \ldots\right]$$

$$\approx \int_0^{\pi/2} d\beta \left[1 - \frac{k^2}{2}\sin^2\beta - \frac{1}{8}k^4\sin^4\beta - \frac{1}{16}k^6\sin^6\beta\right] = \frac{1}{2}\left[\pi - \frac{k^2}{2}\frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1} - \frac{k^4}{8}\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{2} - \frac{k^6}{16}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\pi}{3\cdot 2}\right]$$

$$= \frac{\pi}{2}\left[1 - \frac{k^2}{4} - \frac{3}{8^2}k^4 - \frac{5k^6}{16^2}\right]$$

$$L = \mu_0\sqrt{ab}\left[\left(\frac{2}{k} - k\right)\frac{\pi}{2}\left(1 + \frac{k^2}{4} + \left(\frac{3}{8}\right)^2 k^4 + \underbrace{\left(\frac{5}{16}\right)^2 k^6}_{\text{не треба}}\right) - \frac{2}{k}\frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3}{8^2}k^4 - \underbrace{\frac{5k^6}{16^2}}_{\text{не треба}}\right)\right]$$

$$= \mu_0\sqrt{ab}\,\frac{\pi}{2}\left[\cancel{\frac{2}{k}} + \cancel{\frac{k}{2}} + \frac{9}{32}k^3 + \frac{25}{16\cdot 8}k^5 - \cancel{k} - \frac{k^3}{4} - \frac{9}{64}k^5 - \frac{25}{16^2}k^7 - \cancel{\frac{2}{k}} + \cancel{\frac{k}{2}} + \frac{3}{32}k^3 + \frac{5}{16\cdot 8}k^5\right]$$

$$= \mu_0\sqrt{ab}\,\frac{\pi}{2}\,\frac{1}{8}k^3 = \mu_0\sqrt{ab}\,\frac{\pi}{16}\,\frac{4ab}{h^2}\,\frac{2\sqrt{ab}}{h} = \boxed{\frac{\mu_0\pi a^2 b^2}{2h^3}}$$

Лакше, одмах апроксимирати:

$$L = \mu_0 ab\int_0^\pi \frac{\cos\alpha\, d\alpha}{(h^2+a^2+b^2-2ab\cos\alpha)^{1/2}} \underset{h\to\infty}{\approx} \frac{\mu_0 ab}{h}\int_0^\pi \frac{\cos\alpha\, d\alpha}{\sqrt{1+\frac{a^2+b^2-2ab\cos\alpha}{h^2}}} \approx$$

$$\approx \frac{\mu_0 ab}{h}\int_0^\pi \cos\alpha\left(1 - \frac{1}{2}\frac{\cancel{a^2}+\cancel{b^2}-2ab\cos\alpha}{h^2}\right)d\alpha = \frac{\mu_0 ab}{h}\frac{ab}{h^2}\int_0^\pi \cos^2\alpha\, d\alpha = \boxed{\frac{\mu_0 a^2 b^2}{h^3}\frac{\pi}{2}}$$